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The Frobenius-Perron Operator as a Product of Two Operators

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Abstract—We show that the Frobenius-Perron operator $P_S : L^1(X) \to L^1(X)$ associated with a nonsingular transformation $S : X \to X$ on a σ-finite measure space X is the product of an isometry and a weak contraction.

Keywords—Frobenius-Perron operator, Conditional expectation.

1. INTRODUCTION

Let (X, Σ, μ) be a σ-finite measure space, and let $S : X \to X$ be a nonsingular transformation; i.e., S is measurable and $\mu(S^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. In ergodic theory, the operator $P_S : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ defined implicitly by

$$\int_A P_S f \, d\mu = \int_{S^{-1}(A)} f \, d\mu \tag{1}$$

is called the Frobenius-Perron operator associated with S.

Many problems in physical sciences are related to the problem of the existence of absolutely continuous invariant measures [1]. It is obvious from (1) that for $f \in L^1(X, \Sigma, \mu)$, the measure μ_f defined by

$$\mu_f(A) = \int_A f \, d\mu \qquad \text{for all} \quad A \in \Sigma,$$

which is absolutely continuous with respect to μ, is invariant under S if and only if f is a fixed point of P_S. Here, the invariance of the measure μ_f (under S) means that $\mu_f(S^{-1}(A)) = \mu_f(A)$ for every measurable set A. Hence, the problem of the existence of an absolutely continuous invariant measure for a nonsingular transformation is equivalent to the fixed point problem of the corresponding Frobenius-Perron operator.

Recently, some new theoretical analysis of Frobenius-Perron operators have been given in [2–4]. In this note we show that the Frobenius-Perron operator can be viewed as the product of an isometry and a weak contraction, using the concept of conditional expectation in probability theory. We first review a new definition of P_S which was motivated and developed in [4]. Then we prove our main result.

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2. FROBENIUS-PERRON OPERATORS

We give an equivalent definition of the Frobenius-Perron operator in this section for later use. Throughout the paper we assume that the measure space $(X, S^{-1}\Sigma, \mu)$ is σ-finite as well as the measure space (X, Σ, μ).

First of all, the assumption that $S : X \to X$ is a nonsingular transformation means that the measure $\mu \circ S^{-1}$ is absolutely continuous with respect to the measure μ. By the Radon-Nikodym theorem, there is a unique nonnegative measurable function h such that

$$\mu \circ S^{-1}(A) = \int_A h \, d\mu, \qquad \forall A \in \Sigma;$$

h is called the Radon-Nikodym derivative of $\mu \circ S^{-1}$ with respect to μ, and is denoted by $h = \frac{d\mu \circ S^{-1}}{d\mu}$. We note that $h \in L^1(X, \Sigma, \mu)$ if and only if $\mu \circ S^{-1}$ is a finite measure. Also the assumption of the σ-finiteness of $(X, S^{-1}\Sigma, \mu)$ is equivalent to the condition $h(x) < \infty$ a.e. μ.

Given any $f \in L^1(X, \Sigma, \mu)$, by the Radon-Nikodym theorem, there exists a unique function $Ef \in L^1(X, S^{-1}\Sigma, \mu)$ such that

$$\int_B Ef \, d\mu = \int_B f \, d\mu, \qquad \forall B \in S^{-1}\Sigma.$$

Explicitly, $Ef = \frac{d\mu_f|_{S^{-1}\Sigma}}{d\mu|_{S^{-1}\Sigma}}$, where $d\mu_f = f \, d\mu$. Since $L^1(X, S^{-1}\Sigma, \mu) \subset L^1(X, \Sigma, \mu)$, we obtain a linear operator $E : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ which is called the *conditional expectation* associated with the σ-subalgebra $S^{-1}\Sigma$. Note that the operator E is a bounded projection with norm 1, and so the range $R(E)$ is closed in $L^1(X, \Sigma, \mu)$. See [5] for further details regarding E. The following lemma was proved in [4].

LEMMA 2.1. *For any $S^{-1}\Sigma$-measurable function $f \in L^1(X, \Sigma, \mu)$, there exists a unique $g \in L^1(X, \Sigma, \mu \circ S^{-1})$ such that $g \circ S = f$. Moreover, $\|g\|_{\mu \circ S^{-1}} = \|f\|_\mu$, where $\|g\|_{\mu \circ S^{-1}} = \int_X |g| \, d\mu \circ S^{-1}$ and $\|f\|_\mu = \int_X |f| \, d\mu$.*

COROLLARY 2.1. *For any $f \in L^1(X, \Sigma, \mu)$, there exists a unique $g \in L^1(X, \Sigma, \mu \circ S^{-1})$ such that $g \circ S = Ef$. Furthermore, $\|g\|_{\mu \circ S^{-1}} = \|Ef\|_\mu$.*

The proof of the following result was given by Ding and Hornor [4].

THEOREM 2.1. *The Frobenius-Perron operator $P_S : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ can be defined as*

$$P_S f(x) = h(x) \left(Ef \circ S^{-1}(x) \right),$$

where $(Ef) \circ S^{-1}$ is the unique $g \in L^1(X, \Sigma, \mu \circ S^{-1})$ such that $g \circ S = Ef$ as in Corollary 2.1.

3. P_S AS A PRODUCT

In this section, we show that the Frobenius-Perron operator is the product of an isometry and a weak contraction. First we list some properties h in the following proposition. Let $\operatorname{supp} h = \{x \in X : h(x) \neq 0\}$.

PROPOSITION 3.1. $\operatorname{supp} h = X$ *if and only if the two measures μ and $\mu \circ S^{-1}$ are equivalent. Hence, $P_S : L'(X, \Sigma, \mu) \to L'(X, \Sigma, \mu)$ is surjective if and only if $\mu(X - \operatorname{supp} h) = \emptyset$.*

Now we define $M : L^1(X, \Sigma, \mu \circ S^{-1}) \to L^1(X, \Sigma, \mu)$ and $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu \circ S^{-1})$ by

$$Mg = gh, \qquad \forall g \in L^1\left(X, \Sigma, \mu \circ S^{-1}\right)$$

and

$$Tf = (Ef) \circ S^{-1}, \qquad \forall f \in L^1(X, \Sigma, \mu),$$

respectively. M is a usual multiplication operator. Note that $P_S = MT$. Thus, P_S is actually the product of two linear operators.

LEMMA 3.1. *M is isometric and $R(M)$ is isomorphic to $L^1(\operatorname{supp} h, \Sigma \cap \operatorname{supp} h, \mu)$. In particular, if $\operatorname{supp} h = X$, then $M : L^1(X, \Sigma, \mu \circ S^{-1} \to L^1(X, \Sigma, \mu)$ is isomorphic.*

PROOF. Since

$$\|Mg\|_\mu = \int_X |gh|\, d\mu = \int_X |g|\, d\mu \circ S^{-1} = \|g\|_{\mu \circ S^{-1}},$$

M is an isometry. Now it is easy to see that

$$R(M) = \{f \in L^1(X, \Sigma, \mu) : f(x) = 0, x \in X - \operatorname{supp} h\}$$
$$\cong L^1(\operatorname{supp} h, \Sigma \cap \operatorname{supp} h, \mu). \quad \blacksquare$$

LEMMA 3.2. *T is a weak contraction; that is, $\|Tf\|_{\mu \circ S^{-1}} \leq \|f\|_\mu$ for all $f \in L^1(X, \Sigma, \mu)$. Moreover, if $Ef = f$ for some $f \in L^1(X, \Sigma, \mu)$, then $\|Tf\|_{\mu \circ S^{-1}} = \|f\|_\mu$. Thus, $Ef = f$ for all $f \in L^1(X, \Sigma, \mu)$ implies that T is isometric.*

PROOF. The first conclusion comes from Corollary 2.1, since $\|Ef\|_\mu \leq \|f\|_\mu$. If $Ef = f$ for some $f \in L^1(X, \Sigma, \mu)$, then $\|Tf\|_{\mu \circ S^{-1}} = \|Ef\|_\mu = \|f\|_\mu$. So the second conclusion follows. $\blacksquare$

Combining Lemmas 3.1 and 3.2, we obtain the following decomposition result immediately.

THEOREM 3.1. *$P_S = MT$, where $M : L^1(X, \Sigma, \mu \circ S^{-1}) \to L^1(X, \Sigma, \mu)$ is an isometry, and $T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu \circ S^{-1})$ is a weak contraction.*

COROLLARY 3.1. *Let $f \in L^1(X, \Sigma, \mu)$. If $Ef = f$, then $\|P_S f\|_\mu = \|f\|_\mu$. Thus, $R(E) = L^1(X, \Sigma, \mu)$ implies that $P_S : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is isometric.*

PROOF. Since $P_S = MT$ and M is isometric, if $Ef = f$, then $\|P_S f\|_\mu = \|MTf\|_\mu = \|Tf\|_{\mu \circ S^{-1}} = \|f\|_\mu$. $\blacksquare$

COROLLARY 3.2. *$L^1(X, \Sigma, \mu) = N(P_S) \oplus R(E)$ and $P_S|_{R(E)} : R(E) \to L^1(X, \Sigma, \mu)$ is isometric.*

PROOF. Since $E : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is a positive projection with norm 1, we have $L^1(X, \Sigma, \mu) = N(E) \oplus R(E)$.

Since $P_S = MT$ and M is isometric, $N(P_S) = N(T)$. And since $Tf = (Ef) \circ S^{-1}$ and $\|Tf\|_{\mu \circ S^{-1}} = \|Ef\|_\mu$, we have $N(T) = N(E)$. Thus $N(E) = N(P_S)$. Hence,

$$L^1(X, \Sigma, \mu) = N(P_S) \oplus R(E).$$

The last conclusion follows from Corollary 3.1. $\blacksquare$

Although P_S is not an isometry in general, we have the following proposition.

PROPOSITION 3.2. *$\|P_S f\|_\mu = \|Ef\|_\mu$ for all $f \in L^1(X, \Sigma, \mu)$.*

4. CONCLUSIONS

Using conditional expectation, we expressed the Frobenius-Perron operator P_S as the product of an isometry and a weak contraction. Thus, P_S is isometric if and only if the weak contraction is actually an isometry. With this new approach, many properties of P_S depend on those of the weak contraction T which itself depends on the conditional expectation operator E.

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